

KUMAR'S CRITERION MODULO p

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ABSTRACT. We prove that equivariant multiplicities may be used to determine whether attractive fixed points on T -varieties are p -smooth. This gives a combinatorial criterion for the determination of the p -smooth locus of Schubert varieties for all primes p .

1. INTRODUCTION

Let X be an n -dimensional complex algebraic variety equipped with its classical topology and let p be a prime number. A point $x \in X$ is p -smooth if one has an isomorphism

$$H^\bullet(X, X \setminus \{x\}; \mathbb{F}_p) \cong H^\bullet(\mathbb{C}^n, \mathbb{C}^n \setminus \{0\}; \mathbb{F}_p).$$

The p -smooth locus is the largest open subset of X consisting of p -smooth points. Similarly one defines \mathbb{Z} -smooth and *rationaly smooth* by replacing the \mathbb{F}_p -coefficients above by \mathbb{Z} and \mathbb{Q} respectively. One has inclusions [FW, Section 8.1]

$$\begin{array}{ccccccc} \text{smooth} & & \mathbb{Z}\text{-smooth} & & p\text{-smooth} & & \text{rationally} \\ \text{locus} & \subset & \text{locus} & \subset & \text{locus} & \subset & \text{smooth locus} \end{array}$$

all of which are strict in general. (These inclusions are already strict for Kleinian surface singularities: all such singularities are rationally smooth (being finite quotient singularities), however a singularity of type A_n is p -smooth if and only if $p \nmid n+1$ and a singularity of type E_8 is \mathbb{Z} -smooth.)

The variety X is p -smooth if and only if the constant sheaf on X with coefficients in \mathbb{F}_p is Verdier self-dual (up to a shift). It follows that if X is p -smooth and compact, then Poincaré duality holds for the cohomology of X with coefficients in \mathbb{F}_p (and similarly, if X is \mathbb{Z} - or rationally smooth; in the case of \mathbb{Z} , it is a derived duality).

As is clear from the definition, the notion of p -smoothness is topological in nature. In general it seems difficult to decide whether a point $x \in X$ is p -smooth. In this paper we give a combinatorial criterion which often enables one to decide whether an isolated fixed point of a T -variety is p -smooth.

Let T denote an algebraic torus, let S denote the symmetric algebra on the character lattice of T , and let Q denote its fraction field. We view S as a graded algebra with the characters of T in degree 2. We assume that X is a normal T -variety and that T has finitely many fixed points on X .

To any fixed point $x \in X^T$ one may associate its “equivariant multiplicity” $e_x X \in Q$ which is obtained by localising the fundamental class in equivariant Borel-Moore homology (see Definition 4.2). It is of the form

$$(1.1) \quad e_x X = \frac{f_x}{\chi_1 \chi_2 \cdots \chi_m}$$

with $f_x \in S$, where $\chi_1, \chi_2, \dots, \chi_m$ are characters of T which occur in the tangent space of X at x . This rational function is homogeneous of degree $-2n$ (recall that we have doubled degrees, and that $n = \dim_{\mathbb{C}} X$).

For example, if $x \in X$ is a smooth point, then

$$e_x X = \frac{1}{\det T_x X} = \frac{1}{\chi_1 \chi_2 \cdots \chi_n}$$

where $\chi_1, \chi_2, \dots, \chi_n$ are all the characters of T on the tangent space of X at x .

Even for singular points $x \in X$ the equivariant multiplicity is often readily computed; indeed, if $\pi : Y \rightarrow X$ is a proper surjective T -equivariant morphism of finite degree d and Y^T is finite then, for $x \in X^T$, we have [Bri97, Lemma 16]

$$(1.2) \quad e_x X = \frac{1}{d} \sum_{\substack{y \in Y^T \\ \pi(y)=x}} e_y Y.$$

It is particularly interesting to consider the case of Schubert varieties in flag varieties for reductive algebraic or Kac-Moody groups G with the action of a maximal torus T . In this case, T -fixed points on the flag variety are in bijection with elements of the Weyl group and (thanks to the existence of Bott-Samelson resolutions) there exist purely algebraic formulas for the equivariant multiplicity in terms of the Weyl group action on the root system.

For Schubert varieties, the rational functions $e_x X$ were first defined in 1987 by Kumar, using the nil-Hecke ring [Kum96]. He showed that they may be used to detect the smooth and rationally smooth loci of Schubert varieties. More precisely, if the fraction in (1.1) is reduced, then

$$x \in X \text{ is smooth} \iff f_x = 1.$$

Moreover, if $U := X \setminus \{x\}$ is rationally smooth then:

$$x \in X \text{ is rationally smooth} \iff f_x \text{ is a constant.}$$

This result was generalised to cover T -varieties with isolated fixed points and finitely many one-dimensional orbits by Arabia [Ara98]. A more general statement due to Brion [Bri97] gives a relative version in terms of fixed points under codimension one subtori.

In the rationally smooth case, it is natural to ask what other geometric or topological significance this numerator might have. In the case of a rationally smooth point in a Schubert variety in a flag variety of a simple algebraic group not of type G_2 , Kumar interpreted the numerator of the equivariant multiplicity as the multiplicity of the point [Kum96, Remark 5.3].

The goal of this paper is to show that equivariant multiplicities may also be used to detect p -smoothness:

Theorem 1.1. *Let X be an affine T -variety with an attractive (hence unique) fixed point x . If $U = X \setminus \{x\}$ is p -smooth and if $H_T^\bullet(U; \mathbb{Z})$ is free of p -torsion, then*

$$x \in X \text{ is } p\text{-smooth} \iff f_x \in \mathbb{Z} \text{ and } p \nmid f_x,$$

where f_x is the numerator of the fraction (1.1), assumed to be reduced.

In the theorem, $H_T^\bullet(U; \mathbb{Z})$ denotes the T -equivariant cohomology of U with coefficients in \mathbb{Z} (see Section 3). Note that by a theorem of Sumihiro [Sum74], any attractive T -fixed point on a normal T -variety has a T -stable affine open neighbourhood, so the requirement that X be affine is mostly harmless.

On the other hand, requiring that $H_T^\bullet(U; \mathbb{Z})$ be free of p -torsion may seem quite restrictive (and of course does not appear in the criterion for rational smoothness). Fortunately this condition always holds for (normal slices in) Schubert varieties, as can be proved [FW] using parity sheaves [JMW] (see Section 7 for a precise statement). Hence we obtain a combinatorial recursive criterion to determine the p -smooth locus of Schubert varieties, refining Kumar's original criterion.

To be more specific, let $G \supset B \supset T$ be a complex reductive algebraic group with a Borel subgroup and maximal torus. The set of T -fixed points on the flag variety G/B may be identified with the Weyl group W , and we have the Bruhat decomposition $G/B = \bigsqcup_{x \in W} BxB/B$, whose closure relation is given by the Bruhat order \leq on W . Given a Schubert variety $X_z := \overline{BzB}/B$, let us denote by $f_{y,z}$ the numerator appearing in the equivariant multiplicity of X_z at the T -fixed point $y \leq z$.

Theorem 1.2. *With the above notation, a T -fixed point x in X_z is p -smooth if and only if for all y in the interval $[x, z]$, the numerator $f_{y,z}$ is an integer and is not divisible by p .*

Taking into account Kumar’s criterion for smoothness, we obtain:

Corollary 1.3. *The smooth and \mathbb{Z} -smooth loci of Schubert varieties coincide.*

We remark that both Theorem 1.2 and Corollary 1.3 also hold for Schubert varieties in Kac-Moody flag varieties.

Dyer gives in [Dye] a detailed combinatorial analysis of equivariant multiplicities for Kac-Moody flag varieties, and derives a criterion, in terms of the Bruhat graph, for a point to be rationally smooth. He also gives explicit formulas (in terms of “generalised binomial coefficients”) for the numerators in this case. Combining Dyer’s results with our main theorem yields:

Theorem 1.4. *Let X be a Schubert variety in a (finite) flag variety G/B for a semi-simple algebraic group G . Then its p -smooth locus is the same as its rationally smooth locus for the following primes p :*

- (1) *all p if G contains only components of types A , D and E ,*
- (2) *$p \neq 2$ if G does not contain a component of type G_2 ,*
- (3) *$p \neq 2, 3$ in general.*

For a fixed Kac-Moody Schubert variety, one may use our main theorem to compare the rationally and p -smooth loci for any p using Dyer’s formulas for the numerators in equivariant multiplicities. However in the infinite family of Schubert varieties occurring in a non-finite Kac-Moody flag there is no uniform bound for the primes dividing the numerators. For example, in the affine flag variety of $SL_2(\mathbb{C})$ all natural numbers appear.

Note that this theorem has been obtained independently in [FW] using moment graph techniques. It answers a question of Dyer about a geometric interpretation of the numerator in the equivariant multiplicity. It also answers in the affirmative a question of Soergel: in [Soe00] he asks whether a rationally smooth Schubert variety in G/B is p -smooth, as long as p is larger than the Coxeter number of G .

It is a result due to Peterson (which is reproved by Dyer in [Dye]) that the smooth and rationally smooth loci of Schubert varieties in simply laced type coincide. Combining Corollary 1.3 with [FW]’s proof of Theorem 1.4, we obtain another independent proof of this result.

Lastly, let us also point out that the above result gives a quick proof of a conjecture of Malkin, Ostrik and Vybornov about non smoothly

equivalent singularities in the affine Grassmannian [MOV05] (see Section 7). The point is that, for a rationally smooth T -fixed point x in a Schubert variety, the set of primes dividing f_x (and hopefully f_x itself, see Conjecture 1.5 below) is an invariant of the singularity up to smooth equivalence.

The proof of Theorem 1.1 is quite straightforward, but makes heavy use of the equivariant sheaves and localisation. A key ingredient is that the universal coefficient theorem (relating cohomology over \mathbb{Q}_p , \mathbb{Z}_p and \mathbb{F}_p) allows one to view certain equivariant cohomology groups with coefficients in \mathbb{Z}_p as lattices inside the cohomology over \mathbb{Q}_p . We then use valuation arguments to deduce the theorem. However, as mentioned before, the fact that we can always apply our main theorem to Schubert varieties relies on the results of [FW].

Finally, let us note that in [JW] several examples are computed. (It was on the basis of these examples that we were led to conjecture Theorem 1.1). In these examples we have observed an even stronger connection which we would like to state as a conjecture. Let N denote a T -invariant affine normal slice to a Bruhat cell BxB/B in the Schubert variety X_z and let $f_{x,z}$ be as above.

Conjecture 1.5. If $U := N \setminus \{x\}$ is smooth and N is rationally smooth, then the order of the torsion subgroup of $H^\bullet(U, \mathbb{Z})$ is $f_{x,z}$.

A local version of the conjecture is the following: if U is p -smooth, then the order of the torsion subgroup of $H^\bullet(U, \mathbb{Z}_p)$ is equal to the p -part of d_x . Also, one could hope that this conjecture is true for an attractive fixed point x in an arbitrary normal affine variety X , assuming that $U := X \setminus \{x\}$ is smooth (resp. p -smooth) and that $H_T^\bullet(U, \mathbb{Z})$ is free (resp. free of p -torsion).

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2. NOTATION

All varieties will be complex algebraic equipped with their classical (metric) topology. We denote by p a prime number, \mathbb{F}_p the finite field with p elements and \mathbb{Z}_p (resp. \mathbb{Q}_p) the p -adic integers (resp. numbers). Throughout, k denotes a ring of coefficients (usually \mathbb{Z} , \mathbb{F}_p , \mathbb{Z}_p or \mathbb{Q}_p). If M is a \mathbb{Z}_p -module we denote by $M_{\mathbb{Q}_p} := M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ its extension of scalars to \mathbb{Q}_p .

3. COHOMOLOGY AND EQUIVARIANT COHOMOLOGY

Given a variety X we denote by $D(X) = D(X; k)$ the derived category of k -sheaves on X . Given $\mathcal{F} \in D(X)$ we denote by $\mathbf{H}^\bullet(X, \mathcal{F})$ its (hyper)cohomology, a graded k -module. If X is acted on by a linear algebraic group G we denote by $D_G(X) = D_G(X; k)$ the equivariant derived category of k -sheaves on X [BL94]. If EG denotes a classifying space for G then $D_G(X; k)$ may be described as the full subcategory of $D^b(X \times_G EG)$ consisting of those objects \mathcal{F} such that $q^*\mathcal{F} \cong p^*\mathcal{G}$ for some $\mathcal{G} \in D(X)$, where p and q denote the projection and quotient morphisms:

$$\begin{array}{ccc} & X \times EG & \\ p \swarrow & & \searrow q \\ X & & X \times_G EG \end{array}$$

We denote by $\text{For} = p_*q^* : D_G(X) \rightarrow D(X)$ the forgetful functor. Given any $\mathcal{F} \in D_G(X)$ we may consider its equivariant cohomology

$$\mathbf{H}_G^\bullet(X, \mathcal{F}) := \mathbf{H}^\bullet(X \times_G EG, \mathcal{F}) \in \mathbf{H}_G^\bullet(\text{pt}) - \text{mod}^{\mathbb{Z}}$$

where $\mathbf{H}_G^\bullet(\text{pt}) - \text{mod}^{\mathbb{Z}}$ denotes the category of graded modules over the graded k -algebra $\mathbf{H}_G^\bullet(\text{pt})$. Throughout we write $\mathbf{H}^\bullet(X, \mathcal{F})$ instead of $\mathbf{H}^\bullet(X, \text{For}(\mathcal{F}))$.

Now suppose that $G = T$ is an algebraic torus. Let $\mathcal{X} = \mathcal{X}^*(T)$ denote its character lattice (a free \mathbb{Z} -module), \mathcal{X}_k the extension of scalars of \mathcal{X} to k and $S = S_k = S(\mathcal{X}_k)$ the symmetric algebra of \mathcal{X}_k . If we view S as a graded algebra with $\deg \mathcal{X}_k = 2$ then we have a canonical isomorphism of graded rings

$$\mathbf{H}_T^\bullet(\text{pt}) = S.$$

As is well known, if $\mathbf{H}_G^\bullet(X, \mathcal{F})$ is a free $\mathbf{H}_G^\bullet(\text{pt})$ -module, then the ordinary cohomology of X with coefficients in $\text{For}(\mathcal{F})$ is obtained from the equivariant cohomology by extension of scalars:

$$\mathbf{H}^\bullet(X, \mathcal{F}) \cong \mathbf{H}_G^\bullet(X, \mathcal{F}) \otimes_{\mathbf{H}_G^\bullet(\text{pt})} k.$$

We will need a mild extension of this result when $G = T$ which we were unable to find in the literature:

Proposition 3.1. *Fix a basis e_1, \dots, e_n of $\mathcal{X} = \mathcal{X}(T)$. Let $\mathcal{F} \in D_T(X)$ and suppose that the images of e_1, \dots, e_n in S give a regular sequence for $\mathbf{H}_T^\bullet(X, \mathcal{F}) \in S - \text{mod}^{\mathbb{Z}}$. Then we have an isomorphism*

$$\mathbf{H}^\bullet(X, \mathcal{F}) \cong \mathbf{H}_T^\bullet(X, \mathcal{F}) \otimes_S k = \mathbf{H}_T^\bullet(X, \mathcal{F}) / (S^+ \mathbf{H}_T^\bullet(X, \mathcal{F}))$$

where $S^+ = \bigoplus_{i>0} S^i$ denotes the augmentation ideal generated by homogeneous elements of strictly positive degree.

Proof. First suppose that $\pi : E \rightarrow B$ is a (topological) \mathbb{C}^* -fibration and that $\mathcal{F} \in D(B)$. We have a spectral sequence

$$(3.1) \quad H^p(\mathbb{C}^*) \otimes H^q(B, \mathcal{F}) \Rightarrow H^{p+q}(E, \pi^* \mathcal{F})$$

with differential induced by multiplication by the Chern class $c_1(\pi)$.

Now suppose that $T = \mathbb{C}^*$ so that $\mathfrak{X}(T) = \mathbb{Z}e$. If we apply this to the \mathbb{C}^* -fibration $q : X \times E\mathbb{C}^* \rightarrow X \times_{\mathbb{C}^*} E\mathbb{C}^*$ then, by considering the pull-back diagram

$$\begin{array}{ccc} X \times E\mathbb{C}^* & \longrightarrow & E\mathbb{C}^* \\ \downarrow q & & \downarrow \\ X \times_{\mathbb{C}^*} E\mathbb{C}^* & \longrightarrow & B\mathbb{C}^* \end{array}$$

we see that the first Chern class of q acts on $H_T^\bullet(X, \mathcal{F})$ as multiplication by the image of e in S . Multiplication by e is injective on $H_T^\bullet(X, \mathcal{F})$ because (e) is a regular sequence and it follows that

$$H^\bullet(X, \mathcal{F}) = H^\bullet(X \times E\mathbb{C}^*, q^* \mathcal{F}) = H_{\mathbb{C}^*}^\bullet(X, \mathcal{F}) / (e \cdot H_{\mathbb{C}^*}^\bullet(X, \mathcal{F})).$$

We now turn to the general case. Let $T = \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ be the splitting of T corresponding to the basis (e_1, \dots, e_n) of \mathfrak{X} and let T_i denote the subtorus consisting of the last i copies of \mathbb{C}^* . In other words $T_i = \ker e_1 \cap \cdots \cap \ker e_{n-i}$. Let q_i denote the quotient maps

$$X \times ET \xrightarrow{q_0} X \times_{T_1} ET \xrightarrow{q_1} X \times_{T_2} ET \xrightarrow{q_2} \cdots \xrightarrow{q_{n-2}} X \times_{T_{n-1}} ET \xrightarrow{q_{n-1}} X \times_T ET$$

and set

$$H_i := H^\bullet(X \times_{T_i} ET, q_i^* q_{i+1}^* \cdots q_{n-1}^* \mathcal{F}) \text{ and } H_n := H_T^\bullet(X, \mathcal{F}).$$

By induction, our regular sequence assumption and the spectral sequence (3.1) we have

$$H_i = H_{i+1} / e_i H_{i+1} = H_n / (e_1, \dots, e_i) H_n.$$

Hence

$$H^\bullet(X, \mathcal{F}) = H^\bullet(X \times ET, q^* \mathcal{F}) = H_0 = H_n / (e_1, \dots, e_n) H_n$$

as claimed. \square

4. EQUIVARIANT MULTIPLICITIES

In this section T denotes a complex torus and X is an irreducible n -dimensional T -variety. In this section we always take equivariant cohomology with coefficients in $k = \mathbb{Z}$. Given a T -variety Y , its equivariant constant and dualising sheaves are denoted \underline{k}_Y and ω_Y respectively. We have $H_T^{-m}(Y; \omega_Y) = H_m^T(Y)$ where $H_\bullet^T(Y)$ denotes equivariant Borel-Moore homology. Although we never make use of this isomorphism, it may provide an intuitive aid for the reader below.

We first recall the definition of the equivariant canonical class of X . Let X^{reg} denote the smooth locus of X , then X^{reg} has a canonical orientation, and hence a canonical class $\mu_{X^{\text{reg}}} \in H_T^{-2n}(X^{\text{reg}}, \omega_{X^{\text{reg}}})$. It is straightforward to see that the restriction map

$$r : H_T^{-2n}(X, \omega_X) \rightarrow H_T^{-2n}(X^{\text{reg}}, \omega_{X^{\text{reg}}})$$

is an isomorphism.

Definition 4.1. The *equivariant canonical class* $\mu_X \in H_T^{-2n}(X, \omega_X)$ is defined to be the inverse image of $\mu_{X^{\text{reg}}}$ under the isomorphism r .

For the rest of this section we assume that X^T is finite.

Set $U = X \setminus X^T$ and let i (resp. j) denote the inclusion of X^T (resp. U). Given any $\mathcal{F} \in D_T^t(X)$ we have a standard triangle

$$i_* i^! \mathcal{F} \rightarrow \mathcal{F} \rightarrow j_* j^* \mathcal{F} \rightarrow$$

If we take $\mathcal{F} = \omega_X$ the above triangle may be rewritten as

$$i_* \underline{k}_{X^T} \rightarrow \omega_X \rightarrow j_* \omega_U \rightarrow$$

because $i^!$ and $j^* = j^!$ preserve the dualising sheaf. Taking equivariant (hyper)cohomology we obtain a long exact sequence

$$\dots \rightarrow H_T^m(X^T) \rightarrow H_T^m(X, \omega_X) \rightarrow H_T^m(U, \omega_U) \rightarrow H_T^{m+1}(X^T) \rightarrow \dots$$

Standard arguments (see e.g. [Bri97, FW]) show that $H_T^\bullet(U, \omega_U)$ is a torsion module over S . Since $H_T^\bullet(X^T)$ is a free S -module (of rank $|X^T|$), the above long exact sequence is in fact a short exact sequence of S -modules:

$$0 \rightarrow H_T^m(X^T) \rightarrow H_T^m(X, \omega_X) \rightarrow H_T^m(U, \omega_U) \rightarrow 0$$

and if we tensor with Q , the fraction field of S , we obtain an isomorphism

$$i_* : \bigoplus_{x \in X^T} Q = H_T^\bullet(X^T) \otimes_S Q \xrightarrow{\sim} H_T^\bullet(X, \omega_X) \otimes_S Q.$$

Hence we can find rational functions $(e_x X)_{x \in X^T} \in \bigoplus_{x \in X^T} Q$ such that $i_*((e_x X)_{x \in X^T}) = \mu_X \otimes 1$.

Definition 4.2. The *equivariant multiplicity* of $x \in X$ is the rational function $e_x X \in Q$.

For further discussion about properties of the equivariant multiplicity see the papers [Ara98] and [Bri97]. Note that in [Bri97], Brion works instead with equivariant Chow groups, however this may be seen to be equivalent to the above construction using the cycle map from the equivariant Chow group to equivariant Borel-Moore homology [EG98, Section 2.8].

5. FILTRATIONS AND VALUATIONS

Let M be a free \mathbb{Z}_p -module. Then M is a lattice in the \mathbb{Q}_p -vector space $M_{\mathbb{Q}_p} := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} M$, and we have a filtration:

$$\dots \supset p^{-1}M \supset M \supset pM \supset \dots$$

Definition 5.1. The *valuation* $v(m) = v_M(m)$ of $m \in M_{\mathbb{Q}_p}$ relative to M is the greatest $k \in \mathbb{Z}$ such that $m \in p^k M$, or $+\infty$ if $m = 0$.

For example, we have $m \in M$ if and only if $v(m) \geq 0$. For $d \in \mathbb{Q}_p$ and $m \in M_{\mathbb{Q}_p}$, we have

$$v_M(dm) = v_p(d) + v_M(m)$$

and for $m_1, m_2 \in M$ we have

$$v_M(m_1 + m_2) \geq \min(v_M(m_1), v_M(m_2)).$$

Now suppose that $S = S(\mathcal{X})$ is the symmetric algebra over \mathbb{Z}_p on a free \mathbb{Z} -module \mathcal{X} . Given $f, g \in S_{\mathbb{Q}_p}$ it is straightforward to check that

$$v_S(fg) = v_S(f) + v_S(g).$$

It follows that, if M is a free S -module, then

$$v_M(fm) = v_S(f) + v_M(m)$$

for $m \in M_{\mathbb{Q}_p}$ and $f \in S_{\mathbb{Q}_p}$.

6. PROOF OF THEOREM 1.1

Recall that we assume that X is irreducible, affine and n -dimensional and that $x \in X$ is an attractive (hence unique) T -fixed point, and that $U = X \setminus \{x\}$. By Kumar's criterion and the fact that p -smoothness implies rational smoothness, Theorem 1.1 is equivalent to the following:

Theorem 6.1. *Assume that U is p -smooth, that $H_T^\bullet(U; \mathbb{Z})$ is free of p -torsion and that X is rationally smooth, so that the numerator d in the equivariant multiplicity is an integer. Then*

$$p \nmid d \iff x \in X \text{ is } p\text{-smooth}.$$

Throughout this section, we always take coefficients in \mathbb{Z}_p unless otherwise stated. In particular, ω_X (resp. ω_U) denotes the T -equivariant dualising complex with coefficients in \mathbb{Z}_p . In Section 4 we saw the short exact sequence coming from the standard distinguished triangle for the decomposition $X = U \cup \{x\}$:

$$(6.1) \quad 0 \longrightarrow S = H_T^\bullet(\text{pt}) \xrightarrow{\varphi} H_T^\bullet(X, \omega_X) \xrightarrow{r} H_T^\bullet(U, \omega_U) \longrightarrow 0$$

We assume from now on that X is rationally smooth and abbreviate

$$H := H_T^\bullet(X, \omega_X).$$

By Kumar's criterion we can write

$$(6.2) \quad e_x X = \frac{d}{\pi}$$

where $d \in \mathbb{Z}$, π is a product of n characters, and the fraction is assumed to be reduced.

Remark 6.2. In other words, we simplify the fraction until at most one of d and π has positive valuation. We will see in Lemma 6.8 that, in fact, under the assumptions of the theorem only d can have positive valuation.

Remark 6.3. If there is a finite number of one-dimensional orbits, then there are exactly n of them and π is the product of the corresponding characters up to some scalar multiple (which may be needed to simplify the fraction). However, our proof also applies when there is an infinite number of one-dimensional orbits.

Lemma 6.4. *As X is rationally smooth, we have $H_{\mathbb{Q}_p} \cong S_{\mathbb{Q}_p}[2n]$.*

Proof. By definition, X is rationally smooth if and only if $\omega_{X, \mathbb{Q}_p} \cong \underline{\mathbb{Q}}_{p, X}[2n]$. If this is the case then

$$H_{\mathbb{Q}_p} \cong H_T^\bullet(X, \mathbb{Q}_p)[2n] \cong H_T^\bullet(\{x\}, \mathbb{Q}_p)[2n] \cong S_{\mathbb{Q}_p}[2n]$$

where the first isomorphism follows from the universal coefficient theorem, and the second follows because $x \in X$ is an attractive fixed point (and so X retracts equivariantly onto x). \square

Lemma 6.5. *In (6.1) all modules are free over \mathbb{Z}_p .*

Proof. Certainly S is \mathbb{Z}_p -free and $H_T^\bullet(U, \omega_U) \cong H_T^\bullet(U, \mathbb{Z}_p)$ is \mathbb{Z}_p -free by assumption. Hence H is \mathbb{Z}_p -free, being an extension of S and $H_T^\bullet(U, \omega_U)$. \square

Lemma 6.6. *The S -module $H_T^\bullet(U, \omega_U)$ is annihilated by π and in H we have the relation*

$$d \cdot \varphi(1) = \pi \cdot \mu_X.$$

Proof. By the universal coefficient theorem, if we tensor (6.1) over \mathbb{Z}_p with \mathbb{Q}_p we obtain the corresponding short exact sequence with coefficients in \mathbb{Q}_p :

$$(6.3) \quad 0 \longrightarrow S_{\mathbb{Q}_p} \xrightarrow{\varphi} H_T^\bullet(X, \omega_{X, \mathbb{Q}_p}) \xrightarrow{r} H_T^\bullet(U, \omega_{U, \mathbb{Q}_p}) \longrightarrow 0$$

However, X is rationally smooth and hence $H_T^\bullet(X, \omega_{X, \mathbb{Q}_p}) \cong S_{\mathbb{Q}_p}[2n]$ by Lemma 6.4. By the definition of the equivariant multiplicity, (6.3) this short exact sequence has the form:

$$(6.4) \quad 0 \longrightarrow S_{\mathbb{Q}_p} \xrightarrow{\varphi} S_{\mathbb{Q}_p}[2n] \xrightarrow{r} S_{\mathbb{Q}_p}/(\pi)[2n] \longrightarrow 0$$

where $\varphi(1) = \pi/d$.

Now, by Lemma 6.5 all modules in (6.1) are free over \mathbb{Z}_p . Hence we have a commutative diagram with vertical injections

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \xrightarrow{\varphi} & \mathbf{H}_T^\bullet(X, \omega_X) & \xrightarrow{r} & \mathbf{H}_T^\bullet(U, \omega_U) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S_{\mathbb{Q}_p} & \longrightarrow & S_{\mathbb{Q}_p}[2n] & \longrightarrow & S_{\mathbb{Q}_p}/(\pi)[2n] & \longrightarrow & 0 \end{array}$$

We conclude that, in $\mathbf{H} = \mathbf{H}_T^\bullet(X, \omega_X)$ we have the equation

$$(6.5) \quad d \cdot \varphi(1) = \pi \cdot \mu_X.$$

because this equation holds after extension of scalars to \mathbb{Q}_p , and the above diagram shows that \mathbf{H} injects into the extension of scalars. \square

Lemmas 6.4 and 6.5 show that, if we set $M = S\mu_X$ then M is free as an S -module, and gives a lattice inside $\mathbf{H}_{\mathbb{Q}_p}$. We use the lattice M and apply the terminology of Section 5.

Lemma 6.7. *For all $h \in \mathbf{H}$ we have*

$$v_M(h) \geq -v_p(d).$$

Proof. Given $h \in \mathbf{H}$ then $\pi \cdot h$ is in the kernel of r by Lemma 6.6 and hence we can write $\pi \cdot h = f \cdot \varphi(1) = \frac{1}{d} f \pi \cdot \mu_X$ for some $f \in S$. Applying v_M yields

$$v_S(\pi) + v_M(h) = v_S(f) + v_S(\pi) + v_M(\mu_X) - v_p(d)$$

and hence

$$v_M(h) = v_S(f) - v_p(d)$$

because $v_M(\mu_X) = 0$. The claim now follows because $v_S(f) \geq 0$. \square

In the following lemma, we keep the promise made in Remark 6.2.

Lemma 6.8. *We have $v_S(\pi) = 0$.*

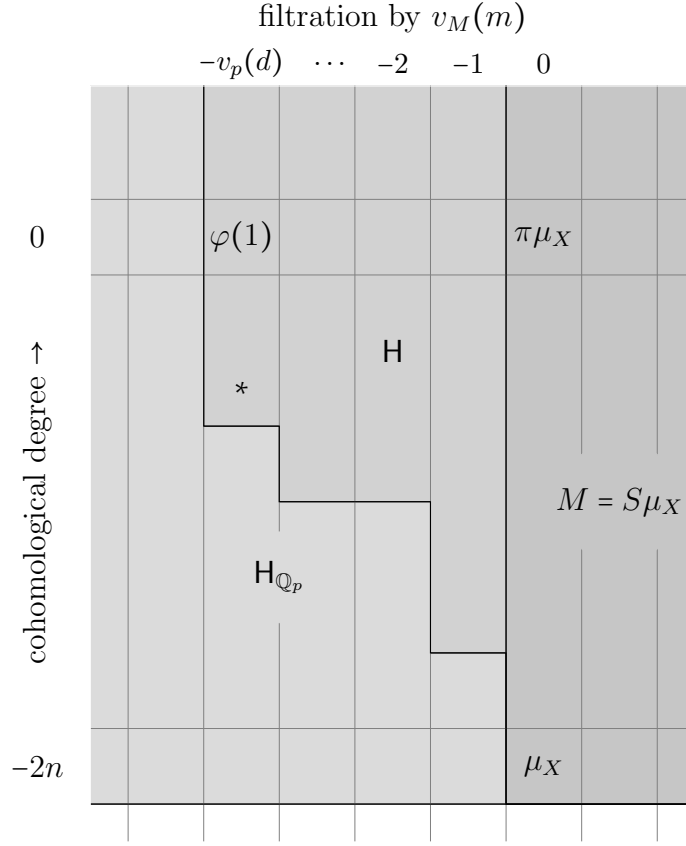
Proof. Because we have assumed that the fraction (6.2) is reduced, if $v_p(d) > 0$ then $v_p(\pi) = 0$. So we may assume that $v_p(d) = 0$. We can write $\pi = p^{v_S(\pi)} \tilde{\pi}$ with $v_S(\tilde{\pi}) = 0$. We have

$$\pi \cdot \mu_X = p^{v_S(\pi)} \tilde{\pi} \cdot \mu_X = d \cdot \varphi(1).$$

As π annihilates $\mathbf{H}_T^\bullet(U, \omega_U)$ we have $p^{v_S(\pi)} r(\tilde{\pi} \cdot \mu_X) = 0$. But by assumption $\mathbf{H}_T^\bullet(U, \omega_U)$ is torsion-free over \mathbb{Z}_p and so $r(\tilde{\pi} \cdot \mu_X) = 0$. Hence $\tilde{\pi} \cdot \mu_X$ is in the image of φ and so $v_M(\tilde{\pi} \cdot \mu_X) \geq v_M(\varphi(1))$. Using that $v_S(\tilde{\pi}) = v_p(d) = 0$ and Lemma 6.6 it follows that

$$0 = v_M(\tilde{\pi} \cdot \mu_X) \geq v_M(\varphi(1)) = v_M(d \cdot \varphi(1)) = v_M(\pi \cdot \mu_X) = v_S(\pi) \geq 0$$

and so $v_S(\pi) = 0$ as claimed. \square

FIGURE 1. The inclusions $M \subset H \subset H_{\mathbb{Q}_p}$.

Remark 6.9. The relation between the modules $M \subset H \subset H_{\mathbb{Q}_p}$ and the induced filtration by valuation is illustrated in Figure 6.

Lemma 6.10. *If (e_1, \dots, e_r) denotes a basis for $\mathcal{X}(T)$ then the images of (e_1, \dots, e_r) in S give a regular sequence for H .*

Proof. Using the inclusion $H \hookrightarrow H_{\mathbb{Q}_p} = S_{\mathbb{Q}_p}$ we see that, for all $1 \leq i \leq r-1$, if $e_{i+1}h \in \langle e_1, \dots, e_i \rangle H$ then $h \in \langle e_1, \dots, e_i \rangle H$ which is the condition for (e_1, \dots, e_r) to give a regular sequence. \square

Lemma 6.11. *X is p -smooth if and only if $H^\bullet(X, \omega_X)$ is torsion-free.*

Proof. In the proof we denote by ω_X^0 the non-equivariant dualising complex on X (with \mathbb{Z}_p coefficients). By definition, X is p -smooth if and only if

$$(6.6) \quad \forall y \in X, \quad \omega_{X,y}^0 \cong \mathbb{Z}_p[2n].$$

By assumption, U is p -smooth so this holds for all $y \neq x$ in X . By a standard argument (see for example the attractive Proposition 2.2 of [FW]) we know that

$$H^\bullet(\omega_{X,x}^0) \cong H^\bullet(X, \omega_X^0).$$

Because X is assumed to be rationally smooth, we know that the free part of $H^\bullet(\omega_{X,x}^0)$ is concentrated in degree $-2n$ where it is of rank one. Hence we have (6.6) if and only if $H^\bullet(X, \omega_X^0)$ is torsion-free. \square

Proof of Theorem 6.1. By Proposition 3.1 and Lemma 6.10 we have

$$H^\bullet(X, \omega_X) = H/(S^+H).$$

By Lemma 6.11 X is p -smooth if and only if $H/(S^+H)$ is torsion-free.

Now choose $m \in H^i$ and let \overline{m} denote its class in $H/(S^+H)$. Then $v_M(dm) = v_p(d) + v_M(m) \geq 0$ by Lemma 6.7. Hence $dm \in S\mu_X$. In other words, $d\overline{m} = 0$ unless $i = -2n$. It follows that multiplication by d annihilates the torsion in $H/(S^+H)$. If $p \nmid d$ then multiplication by d is an automorphism of $H/(S^+H)$, and hence $H/(S^+H)$ is torsion-free.

Now let us assume that $p \mid d$, i.e. $v_p(d) > 0$. Let $f \in S$ be a homogeneous element of maximal degree such that $\varphi(1) = fh$ for some $h \in H$. Note that $h \notin S^+H$.¹ By Lemma 6.7 and Lemma 6.6, we have

$$-v_p(d) = v_M(\varphi(1)) = v_S(f) + v_M(h) \geq 0 - v_p(d)$$

hence we have both equalities $v_S(f) = 0$ and $v_M(h) = -v_p(d)$. In particular, $h \notin H^{-2n} = \mathbb{Z}_p\mu_X$. Now, $v_M(d \cdot h) = 0$ so $dh \in M = S\mu_X$. By the previous observation, we actually have $dh \in S^+\mu_X \subset S^+H$. So the image \overline{h} of h in H/S^+H is non-zero and torsion. Since H/S^+H is not torsion-free, X is not p -smooth. \square

7. THE CASE OF SCHUBERT VARIETIES

In this section, G denotes a connected reductive complex algebraic group, and we make a choice $G \supset B \supset T$ of a Borel subgroup and a maximal torus. Let $X = G/B$ be the flag variety and W the Weyl group. For $w \in W$, let $C_w := BwB/B$ be the corresponding Bruhat cell (which is an affine space of dimension equal to the length of w), with closure the Schubert variety $X_w = \overline{C}_w$. We have the Bruhat decomposition

$$X = \bigsqcup_{w \in W} C_w.$$

(More generally, we could take X be a partial flag variety for a Kac-Moody group, with appropriate modifications.)

¹A clue as to the position of h in our diagram is given by an asterix.

Recall from [KL80] (or [Kum02] in the Kac-Moody case) that, for any elements $x \leq w$ in W , we can find an affine neighbourhood \tilde{N} of C_x in X_w , a closed subset N in \tilde{N} and an isomorphism

$$C_x \times N \xrightarrow{\sim} \tilde{N} \subset X_w.$$

We will use the following result which is proved in [FW, Corollary 8.9].

Proposition 7.1. *Suppose that N is p -smooth. Then $H_T^\bullet(N \setminus \{x\}, \mathbb{Z}_p)$ is torsion-free.*

Hence we can apply our main theorem 1.1 in the case of Schubert varieties. Theorem 1.2 and Corollary 1.3 follow immediately. Theorem 1.4 follows from [Dye, Corollary 3.5] which shows that the numerator of the equivariant multiplicity at a rationally smooth point in a Schubert variety for a finite flag variety is always of the form $2^a 3^b$, with b possibly non-zero only if G contains a component of type G_2 , and $a = b = 0$ in simply-laced types.

Lastly, in [MOV05], Malkin, Ostrik and Vybornov study minimal degenerations in affine Grassmannians up to smooth equivalence. They find the following possibilities: simple singularities of type A in the codimension 2 case, minimal nilpotent singularities (the singularity of 0 in the closure of the minimal nilpotent orbit in a simple Lie algebra), and some presumably new singularities which they call quasi-minimal of type ac_n (of codimension $2n$, arising in type C_n), and ag_2 and cg_2 (of codimension 4, arising in type G_2). They compute their local (rational) intersection cohomology using the Kazhdan-Lusztig algorithm, and their equivariant multiplicities. It turns out that ac_n (resp. ag_2 , cg_2) has the same local rational intersection cohomology as a minimal singularity of type a_n (resp. a_2 , c_2). Malkin, Ostrik and Vybornov conjectured that the pairs (ac_n, a_n) , (ag_2, a_2) , (ac_2, ag_2) and (c_2, cg_2) are not smoothly equivalent. Among those, only the last one involves rationally smooth singularities. The numerator for g_2 is 18, whereas the numerator for cg_2 is 27, so cg_2 is 2-smooth and g_2 is not. Hence these singularities are not smoothly equivalent. To prove their conjecture in the other cases, one needs either to do a more involved calculation or to use the geometric Satake correspondence [JW].

8. A ZOO OF (RATIONALLY SMOOTH) POINTS

We conclude with some examples of rationally smooth attractive fixed points, their equivariant multiplicities and the cohomology of the complement. It was based on these and other examples that the authors were led to believe that something like the main theorem 1.1 must hold.

The reader can also verify that the more precise Conjecture 1.5 holds in all of these cases.

8.1. Smooth points. Let $x = 0 \in X = \mathbb{C}^n$ with a torus T acting linearly with characters χ_1, \dots, χ_n . Then

$$\dim X = n,$$

$$e_x X = \frac{1}{\chi_1 \chi_2 \cdots \chi_n}.$$

The cohomology $H^\bullet(X \setminus \{x\}; \mathbb{Z})$ is given by:

$$\begin{array}{cccccc} H^0 & H^1 & H^2 & \cdots & H^{2n-2} & H^{2n-1} \\ \hline \mathbb{Z} & 0 & 0 & \cdots & 0 & \mathbb{Z} \end{array}$$

8.2. Kleinian singularities of type A . Let $x = 0 \in X = \mathbb{C}^2/\mu_{n+1}$ where μ_{n+1} denotes the group of $n+1$ roots of unity acting via $\mu \cdot (x, y) = (\mu x, \mu^{-1} y)$. Then X is a Kleinian surface singularity of type A_n . We can embed X as the locus of (u, v, w) in \mathbb{C}^3 such that $uv = w^{n+1}$. Then X has an attractive $T = (\mathbb{C}^*)^2$ action given by

$$(\lambda_1, \lambda_2) \cdot (u, v, w) = (\lambda_1 \lambda_2^n u, \lambda_1^{-1} \lambda_2 v, \lambda_2 w).$$

The map $\pi : X \rightarrow \mathbb{C}^2$ induced by the projection $(u, v, w) \mapsto (u, v)$ is finite of degree $n+1$. Applying (1.2) and the case of a smooth point discussed above yields

$$e_x X = \frac{n+1}{(e_1 + n e_2)(e_2 - e_1)}$$

where e_1 and e_2 are the characters of T given by $e_i(\lambda_1, \lambda_2) = \lambda_i$. The cohomology $H^\bullet(X \setminus \{x\}; \mathbb{Z})$ is given by:

$$\begin{array}{cccc} H^0 & H^1 & H^2 & H^3 \\ \hline \mathbb{Z} & 0 & \mathbb{Z}/(n+1) & \mathbb{Z} \end{array}$$

This calculation follows easily from the discussion in [JMW10, Section 3.4].

8.3. Minimal nilpotent orbit singularities. Let G denote a connected complex simple algebraic group, \mathfrak{g} its Lie algebra, $\mathcal{N} \subset \mathfrak{g}$ its nilpotent cone and $\mathcal{O}_{\min} \subset \mathcal{N}$ the minimal nilpotent orbit. The closure $X = \overline{\mathcal{O}_{\min}}$ is singular, with unique singular point $0 \in \mathfrak{g}$. It turns out that $\overline{\mathcal{O}_{\min}}$ is only rationally smooth in types C_n (including types $C_1 = A_1$ and $C_2 = B_2$) and G_2 . In this section we discuss what our main theorem has to say in these cases.

Let $T \subset G$ denote a maximal torus. Then $X = \overline{\mathcal{O}_{\min}}$ is a $\tilde{T} := T \times \mathbb{C}^*$ variety, where T acts by conjugation and \mathbb{C}^* acts by scaling. We write the characters of $T \times \mathbb{C}^*$ as $\mathcal{X}^*(T) \oplus \mathbb{Z}\delta$ where $\mathcal{X}^*(T)$ denotes

the character lattice of T and δ denotes the identity character of \mathbb{C}^* . Finally, let us denote the set of roots by $\Phi \subset \mathcal{X}^*(T)$, the subset of long roots by $\Phi_{\text{lg}} \subset \Phi$ and the Weyl group by W .

Fix a Borel subgroup B of G containing T and let $\Phi^+ \subset \Phi$ denote the non-trivial characters of T which occur in the Lie algebra of B . One can describe the singularity X as follows [Jut08]: let $\tilde{\alpha}$ denote the highest root with respect to Φ^+ , and let \tilde{I} denote the set of simple roots orthogonal to $\tilde{\alpha}$. The stablizer of $\tilde{\alpha}$ in W is the parabolic subgroup $W_{\tilde{I}}$, and the subgroup of G stabilizing the root subspace $\mathfrak{g}_{\tilde{\alpha}} \subset \mathfrak{g}$ is the parabolic subgroup $P_{\tilde{I}} = BW_{\tilde{I}}B$. Then X is obtained from the line bundle $Y := G \times^{P_{\tilde{I}}} \mathfrak{g}_{\tilde{\alpha}}$ over $G/P_{\tilde{I}}$ by contracting the null section, and the contraction morphism $\pi : Y \rightarrow X$ is a resolution of singularities. The variety Y can be seen as the set of pairs (x, L) where L is a line contained in $\overline{\mathcal{O}}_{\min}$ and x is an element of L .

One may apply (1.2) to the resolution π to find a formula for the equivariant multiplicity valid in any type:

$$(8.1) \quad e_0 \overline{\mathcal{O}}_{\min} = - \sum_{w \in W/W_{\tilde{I}}} \frac{1}{w((\delta + \tilde{\alpha})(\prod_{\alpha \in \Phi^+ \setminus \Phi_{\tilde{I}}^+} \alpha))}$$

Indeed, \tilde{T} -fixed points in $\pi^{-1}(0)$ must be in the null section of Y because they are fixed by \mathbb{C}^* . Now the null section is $G/P_{\tilde{I}}$, and the T -fixed points are in bijection with $W/W_{\tilde{I}}$. One finds the sum of the right-hand-side, the case $w = 1$ corresponding to the line $\mathfrak{g}_{\tilde{\alpha}}$. Since Y is a vector bundle over $G/P_{\tilde{I}}$, the tangent space to Y at this point can be decomposed into the tangent space to $G/P_{\tilde{I}}$ at its base point, whose \tilde{T} -weights are the $-\alpha$ for $\alpha \in \Phi^+ \setminus \Phi_{\tilde{I}}^+$, and the tangent space of the fibre $\mathfrak{g}_{\tilde{\alpha}}$ whose \tilde{T} -weight is $\delta + \tilde{\alpha}$. Note that $\dim G/P_{\tilde{I}} = \dim \mathcal{O}_{\min} - 1$ is odd, hence the global minus sign.

The one-dimensional \tilde{T} -orbits are the long root subspaces. Let now $\varphi : X \rightarrow \bigoplus_{\alpha \in \Phi_{\text{lg}}} \mathfrak{g}_{\alpha} =: V$ be the composition of the inclusion $X \rightarrow \mathfrak{g}$ followed by the natural projection. By the proof of Theorem 18 in [Bri97] this morphism is finite, and in the types C_n and G_2 where X is rationally smooth, it is surjective. In this case, if d denotes its degree, then

$$(8.2) \quad e_0 X = \frac{d}{\prod_{\alpha \in \Phi_{\text{lg}}} (\delta + \alpha)}.$$

We first discuss the case of type C_n . So let $G = \text{Sp}(2n)$ be the symplectic group and \mathfrak{sp}_{2n} its Lie algebra. In this case X is isomorphic to $\mathbb{C}^{2n}/\{\pm 1\}$ (diagonal action) [JMW10, Section 3.3], and the morphism $X \rightarrow Z$ of the last paragraph is identified with $\mathbb{C}^{2n}/\{\pm 1\} \rightarrow$

$\mathbb{C}^{2n}/\{\pm 1\}^{2n} \simeq \mathbb{C}^{2n}$ which is of degree 2^{2n-1} . In this case it is more convenient to apply formula (8.2). We have

$$\dim X = 2n,$$

$$e_0 X = \frac{2^{2n-1}}{\prod_{\alpha \in \Phi_{\text{lg}}} (\delta + \alpha)}.$$

On the other hand, the cohomology $H^\bullet(X \setminus \{0\}; \mathbb{Z})$ is given by:

$$\begin{array}{cccccccccc} H^0 & H^1 & H^2 & H^3 & H^4 & \dots & H^{4n-3} & H^{4n-2} & H^{4n-1} \\ \hline \mathbb{Z} & 0 & \mathbb{Z}/(2) & 0 & \mathbb{Z}/(2) & \dots & 0 & \mathbb{Z}/(2) & \mathbb{Z} \end{array}$$

Now suppose that G is of type G_2 . One case use (8.1) to compute the equivariant multiplicity:

$$\dim X = 6,$$

$$e_0 X = \frac{18}{\prod_{\alpha \in \Phi_{\text{lg}}} (\delta + \alpha)}.$$

The calculation of the cohomology of $X \setminus \{0\} = \mathcal{O}_{\min}$ is performed in [Jut08, Section 3.9]:

$$\begin{array}{cccccccccccc} H^0 & H^1 & H^2 & H^3 & H^4 & H^5 & H^6 & H^7 & H^8 & H^9 & H^{10} & H^{11} \\ \hline \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z}/(3) & 0 & \mathbb{Z}/(2) & 0 & \mathbb{Z}/(3) & 0 & 0 & \mathbb{Z} \end{array}$$

8.4. The quasi-minimal cg_2 singularity. In this example we let G be a simple algebraic group of type G_2 . We use the notation of [MOV05]: \mathcal{G}_G denotes the affine Grassmannian of G , ϖ_1^\vee and ϖ_2^\vee denote the fundamental coweights (which we regard as points of \mathcal{G}_G), $\overline{\mathcal{G}_{\varpi_2^\vee}}$ denotes the Schubert variety indexed by ϖ_2^\vee , and

$$X = (L^{<0} G \cdot \varpi_1^\vee) \cap \overline{\mathcal{G}_{\varpi_2^\vee}}$$

is an affine normal slice to the orbit of ϖ_1^\vee in $\overline{\mathcal{G}_{\varpi_2^\vee}}$. Then X is a \tilde{T} -variety where $\tilde{T} = T \times \mathbb{C}^*$ denotes the extended torus, and ϖ_1^\vee is a attractive \tilde{T} -fixed point.

We have

$$\dim X = 4,$$

$$e_x X = \frac{27}{(\alpha_0 + \alpha_1)(\alpha_0 + \alpha_1 + 3\alpha_2)(2\alpha_0 + 5\alpha_1 + 6\alpha_2)(2\alpha_0 + 5\alpha_1 + 9\alpha_2)}.$$

The cohomology $H^\bullet(X \setminus \{x\}; \mathbb{Z})$ is given by:

$$\begin{array}{cccccccc} H^0 & H^1 & H^2 & H^3 & H^4 & H^5 & H^6 & H^7 \\ \hline \mathbb{Z} & 0 & \mathbb{Z}/(3) & 0 & \mathbb{Z}/(3) & 0 & \mathbb{Z}/(3) & \mathbb{Z} \end{array}$$

(The calculation of the equivariant multiplicity was performed, using Kumar's formula [Kum96], in [MOV05]. The calculation of the cohomology $H^\bullet(X \setminus \{x\}; \mathbb{Z})$ may be performed using moment graph techniques [JW]).

8.5. Other Kleinian singularities. We conclude by discussing Kleinian singularities of types D and E . These examples are intended to convince the reader that once one drops the assumption that $H_T^\bullet(U)$ is torsion-free one cannot hope to have a connection between p -smoothness and the equivariant multiplicity (as in our main theorem).

Recall that if $\Gamma \subset SL_2(\mathbb{C})$ is a finite subgroup then the quotient $X = \mathbb{C}^2/\Gamma$ is a Kleinian singularity. Also X has a unique singular point x given by the image of $0 \in \mathbb{C}^2$. If Γ is cyclic, then X is a Kleinian singularity of type $A_{|\Gamma|-1}$ and our main theorem applies (after appropriate choice of torus action). We have discussed this case above.

However, if Γ is not cyclic then X still admits an attractive \mathbb{C}^* -action. The equation of X in \mathbb{C}^3 as well as the weights of \mathbb{C}^* are given as follows:

$$\begin{aligned} D_n : X^{n-1} + XY^2 + Z^2 &= 0, & \text{weights: } (2, n-2, n-1), \\ E_6 : X^4 + Y^3 + Z^2 &= 0, & \text{weights: } (3, 4, 6), \\ E_7 : X^3Y + Y^3 + Z^2 &= 0, & \text{weights: } (4, 6, 9), \\ E_8 : X^5 + Y^3 + Z^2 &= 0, & \text{weights: } (6, 10, 15). \end{aligned}$$

In each case the projection $(X, Y, Z) \mapsto (X, Y)$ induces a finite surjective map of degree 2. Applying (1.2) one may calculate

$$e_x X = \frac{1}{d\chi^2}$$

where χ denotes the identity character of \mathbb{C}^* , and $d = n-2$ in type D_n , and $d = 6, 12$ and 30 in types E_6 , E_7 and E_8 respectively.

However in [Jut09] the first author has shown that X is p -smooth if and only if p does not divide the index of connection of the corresponding root system. In particular, X is p -smooth if and only if $p \neq 2$ in type D_n , $p \neq 3$ in type E_6 , $p \neq 2$ in type E_7 and X is \mathbb{Z} -smooth in type E_8 . One can check directly that $H_T^\bullet(X \setminus \{x\}, \mathbb{Z})$ has torsion in all cases except E_8 . (Which explains why these examples do not contradict our main theorem!) Hence in these cases there seems to be no relation between those p for which p is not p -smooth and the equivariant multiplicity.

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